Derivation of Bruhn Stiffener Moment of Inertia Equations

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Bruhn Stiffener Moment of Inertia Equations

- Bruhn Chapter C10.10 (1973 edition) has 2 equations for the required moment of inertia for vertical stiffeners to use on non-buckling shear webs:

Equation C10.8

$$I_v = 2.29\frac{d}{t}\left(\frac{Vh}{33E}\right)^{4/3}$$

This also appears in the earliest edition of Bruhn (1942)

Figure C10.9

$$\frac{I_v}{dt^3} = \frac{0.0217}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$$

This is referred to as the "more recent" method

Questions: Where did these come from? What assumptions are they based on?

Bruhn, page C10.7

Part 1: Bruhn Equation C10.8

C10. 10 Stiffener Size to Use with Non-Buckling Web.

A web stiffener is used to decrease the size of web panel; thus when buckling of the web starts, the stiffener tends to keep buckles from extending across the stiffener or causes the sheet to buckle in two panels instead of one. Mr. H. Wagner in a paper presented before a meeting of the A.S.M.E. in 1930 offered the following expression as the required moment of inertia of a stiffener to be used with a shear resistant web.

$$I_V = \frac{2.29d}{t}\left(\frac{V\ h_W}{33\ E}\right)^{4/3} \qquad \text{- - - - - - - - - - (C10.8)}$$

Note: parentheses missing from text

where

I_V = moment of inertia of stiffener

d = center line distance between stiffeners

h_W = depth of web plate

V = vertical shear at section

t = web thickness

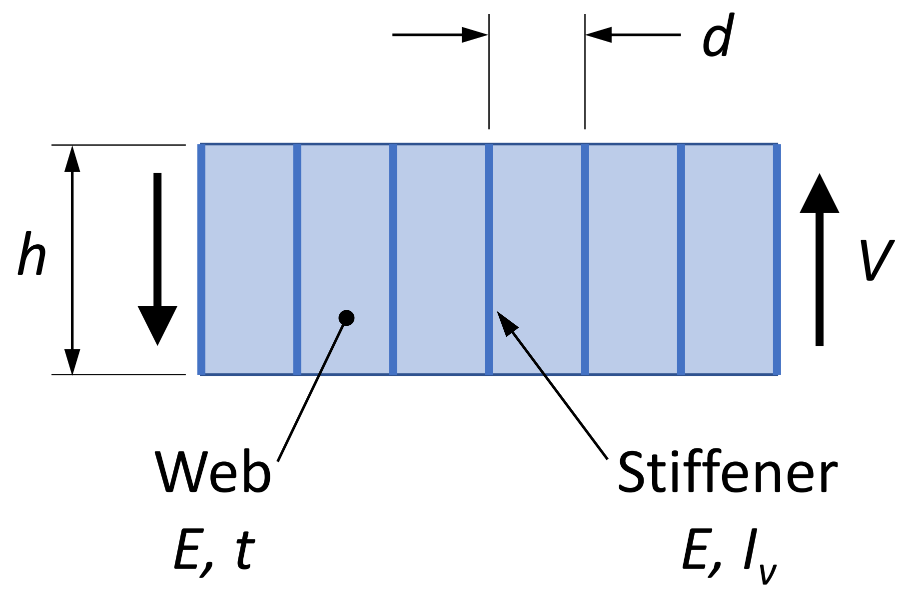
E = modulus of elasticity

Notation

Given a shear web with vertical stiffeners:

d = stiffener spacing (in)
E = modulus of elasticity, web & stiffeners (psi)
h = depth of web (in)
I_v = required moment of inertia (in^4)
K_s = shear buckling coefficient (non-dimensional)
t = web thickness (in)
V = applied vertical shear load (lb)

The shear buckling coefficient K_s will be shown later



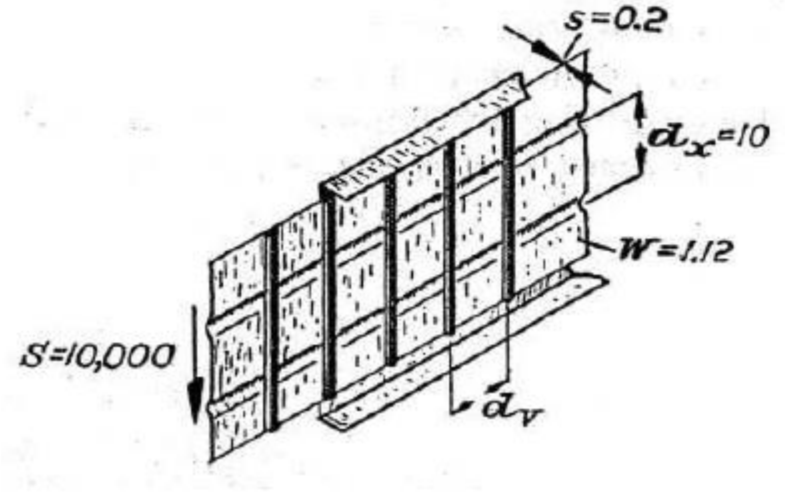
Derivation of Bruhn Equation C10.8

- Bruhn refers to a 1930 paper by Wagner. I assume it is the following paper: *Sheet Metal Airplane Construction*, Herbert Wagner, ASME paper AER-53-18, 1930
- The required stiffener moment of inertia can be derived from the shear buckling equation given by Wagner, but the first question is: where did his buckling equation come from?

<u>Shear Buckling Equation:</u>

$$V = S = \frac{33E}{h}\sqrt[4]{\left(\frac{I_v}{d_v}\right)^3 \frac{I_x}{d_x}}$$

V = S = total shear load at section (lb)



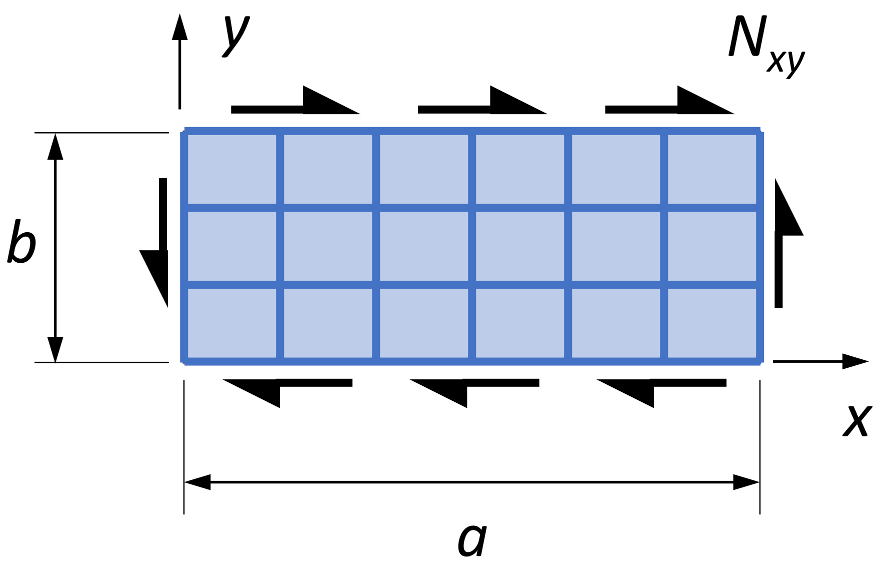
Derivation of Wagner's Shear Buckling Equation

- Refer to NACA-TM-705, *The Critical Shear Load of Rectangular Plates*, Edgar Seydel, April 1933, which gives the shear buckling load of an orthotropic plate as:

$$N_{xycr} = t_{kr} = c_a \frac{\sqrt[4]{D_1 \, D_2{}^3}}{(b/2)^2}$$

where:

t_{kr} = shear buckling load (lb/in)
c_a = shear buckling coefficient (non-dimensional)
D_1 = bending stiffness in x-direction (in-lb)
D_2 = bending stiffness in y-direction (in-lb)
a = long dimension of plate (in)
b = short dimension of plate (in)



Derivation of Wagner's Shear Buckling Equation

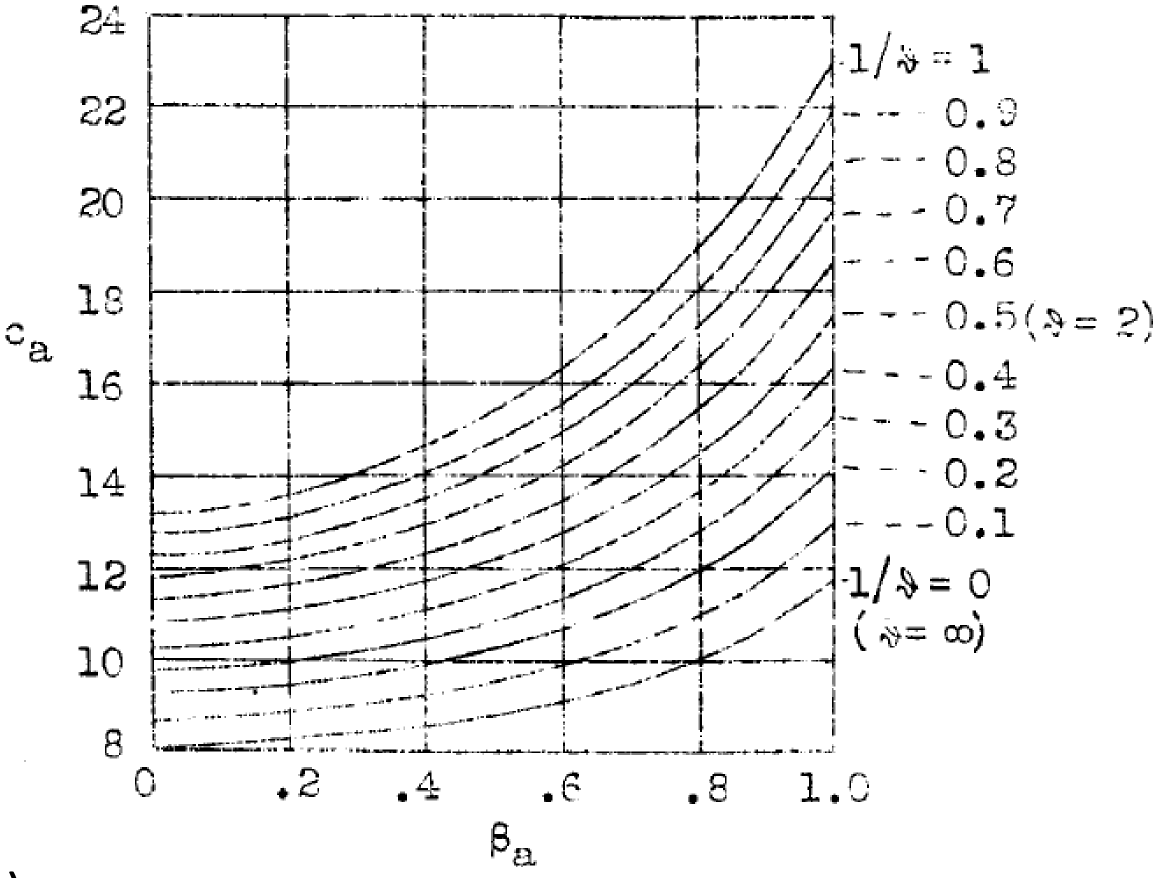
- Seydel's shear buckling coefficient c_a is a function of the effective aspect ratio and the orthotropy parameter defined as:

Effective aspect ratio:

$$\beta_a = \frac{b}{a}\sqrt[4]{\frac{D_1}{D_2}} \quad (\leq 1)^*$$

Orthotropy parameter:

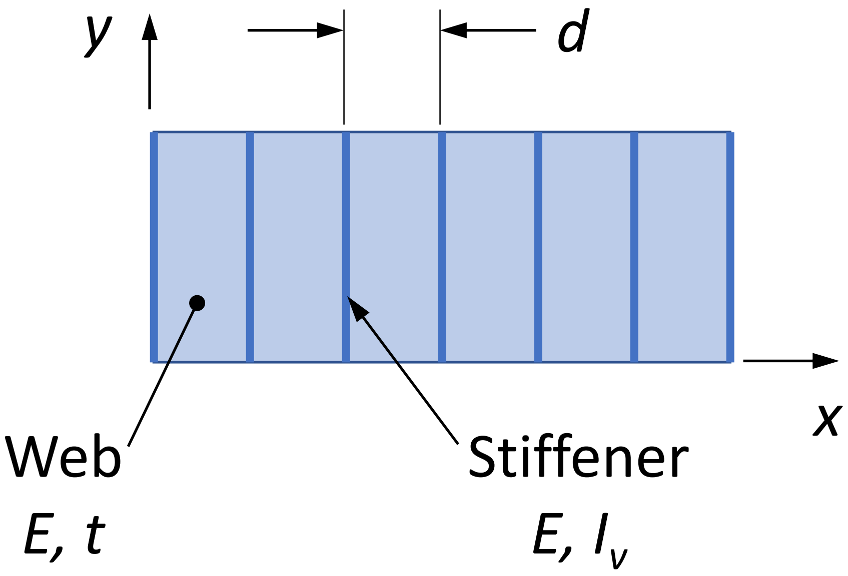
$$\frac{1}{\vartheta} = \frac{D_3}{\sqrt{D_1 D_2}} \qquad (D_3 = D_{12} + 2D_{66})$$



Seydel, Fig. 3

Derivation of Wagner's Shear Buckling Equation

- A stiffened plate can be represented as an orthotropic plate by smearing the stiffnesses as follows:



Assuming stiffeners in y-direction only:

$$D_1 = D = \frac{Et^3}{12(1-\nu^2)} \approx \frac{Et^3}{12} \quad \text{(neglect Poisson's ratio)}$$

$$D_2 = \frac{Et^3}{12(1-\nu^2)} + \frac{EI_v}{d} \approx \frac{EI_v}{d} \quad \text{(plate contribution small)}$$

$$D_3 = D_{12} + 2D_{66} = \nu D + \frac{(1-\nu)}{2}D = D \quad \text{(web only)}$$

Derivation of Wagner’s Shear Buckling Equation

- Using the above plate stiffness terms, the effective aspect ratio can be written as:

$$\beta_a = \frac{b}{a}\sqrt[4]{\frac{D_1}{D_2}} \quad (\leq 1)^*$$

$$\beta_a = \frac{b}{a}\sqrt[4]{\frac{D_1}{D_2}}$$

$a > b$ (long plate)

$D_2 \gg D_1$ (stiffened in y-direction only)

therefore $\boxed{\beta_a \rightarrow 0}$

Derivation of Wagner's Shear Buckling Equation

- Using the above plate stiffness terms, the orthotropy parameter can be written as:

$$\frac{1}{\vartheta} = \frac{D_3}{\sqrt{D_1 \; D_2}}$$

$$\frac{1}{\theta} = \frac{D_3}{\sqrt{D_1 D_2}} \approx \frac{D}{\sqrt{D}\sqrt{EI_v/d}} = \left(\frac{D}{EI_v/d}\right)^{\frac{1}{2}}$$

$$\frac{1}{\theta} = \left(\frac{D}{EI_v/d}\right)^{\frac{1}{2}} = \left(\frac{Et^3/12}{EI_v/d}\right)^{1/2} = \left(\frac{t^3/12}{I_v/d}\right)^{1/2}$$

but $\frac{I_v}{d} \gg \frac{t^3}{12}$ therefore $\boxed{\frac{1}{\theta} \rightarrow 0}$

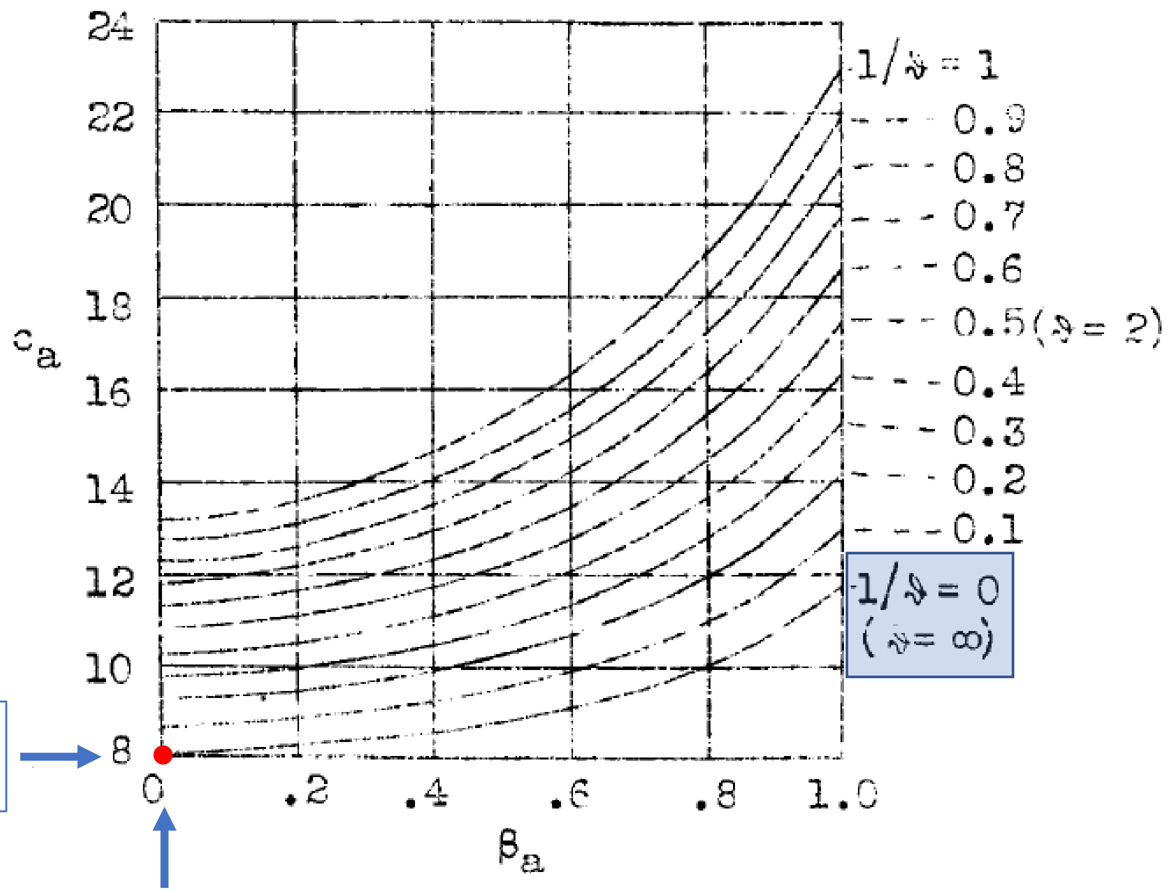
Derivation of Wagner's Shear Buckling Equation

- Now we can get the shear buckling coefficient:

Since: $\boxed{\beta_a \to 0}$ (left hand side of Fig. 3)

$\boxed{\frac{1}{\theta} \to 0}$ (lowest curve in Fig. 3)

therefore $\boxed{\rightarrow c_a = 8.125}$



Seydel Fig. 3

Derivation of Wagner's Shear Buckling Equation

- Substitute terms into Seydel's shear buckling equation to derive Wagner's shear buckling equation:

$$D_1 \approx \frac{Et^3}{12} \quad D_2 \approx \frac{EI_v}{d} \quad c_a = 8.125 \quad \Rightarrow \quad N_{xycr} = t_{kr} = c_a \frac{\sqrt[4]{D_1 \; D_2^{\,3}}}{(b/2)^2}$$

$$N_{xycr} = \frac{4c_a}{b^2}\left[\frac{Et^3}{12}\left(\frac{EI_v}{d}\right)^3\right]^{1/4}$$

$4c_a = 4(8.125) = 32.5$ (Wagner rounds this to 33)

$$V = S = N_{xycr}\, b = \frac{33}{b}\left[\frac{Et^3}{12}\left(\frac{EI_v}{d}\right)^3\right]^{1/4}$$

Leaving it in this form allows you to account for different moduli of web and stiffeners

Seydel's *b* is Wagner's *h* $\Rightarrow$

$$S = \frac{33}{h}\left[\frac{Et^3}{12}\left(\frac{EI_v}{d}\right)^3\right]^{1/4}$$

$$S = \frac{33E}{h}\sqrt[4]{\left(\frac{I_v}{d_v}\right)^3 \frac{I_x}{d_x}}$$

Derivation of Bruhn Equation C10.8

- Now set global shear buckling load *S* equal to the applied shear load *V* and solve for the required stiffener moment of inertia:

$$S = \frac{33E}{h}\left[\frac{t^3}{12}\left(\frac{I_v}{d}\right)^3\right]^{1/4} = V \quad \Rightarrow \quad I_v = 2.289\frac{d}{t}\left(\frac{Vh}{33E}\right)^{4/3}$$

Bruhn equation C10.8

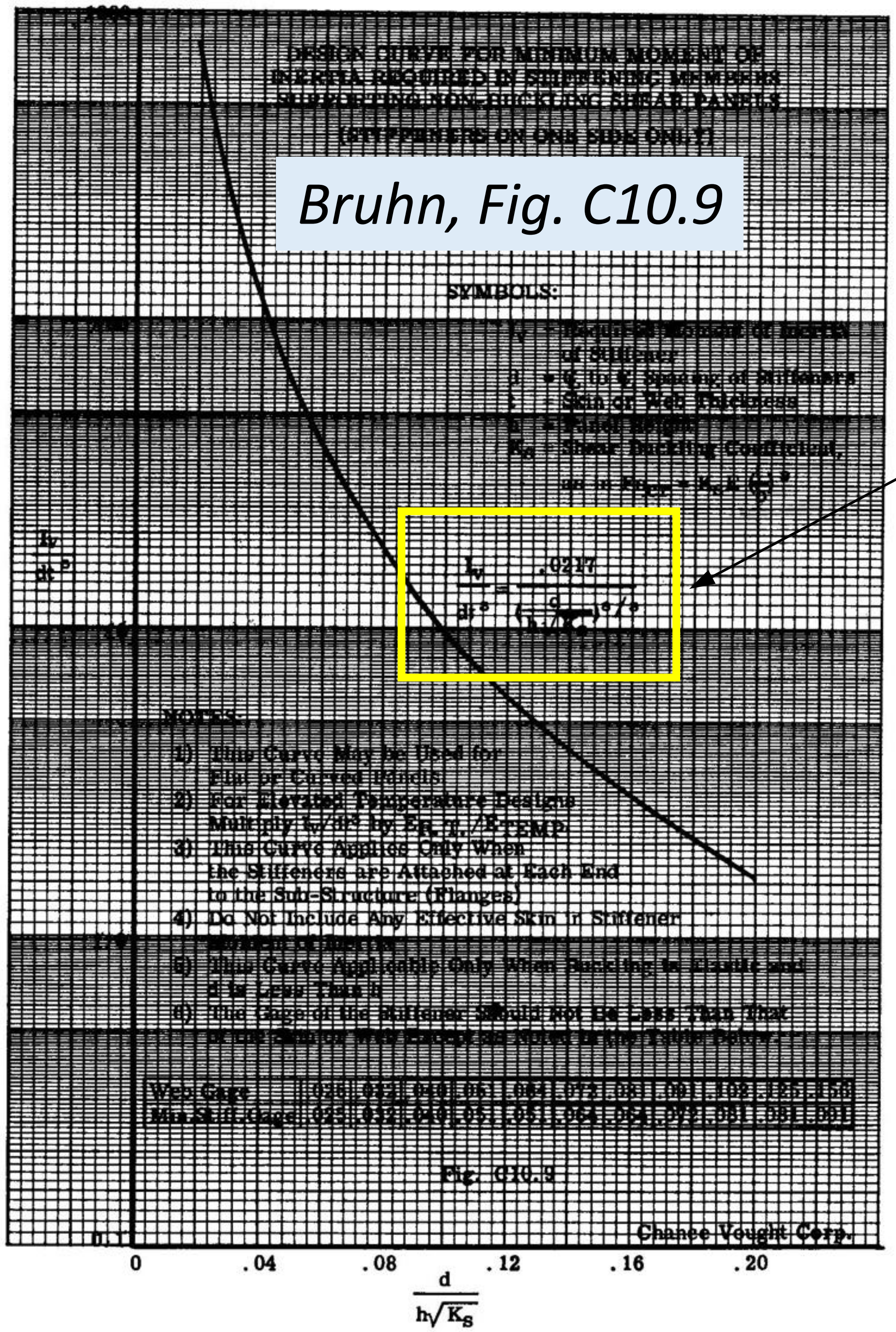
where $2.289 = 12^{1/3}$
(Bruhn rounds this to 2.29)

Recap of Bruhn Equation C10.8

- Starts with shear buckling equation of an orthotropic plate, in which the stiffeners are smeared into the plate stiffness terms, then the stiffener moment of inertia is determined in terms of the other parameters
- Although Bruhn implies this moment of inertia will act as a panel breaker, such behavior was not really part of the derivation. The derivation was for the required stiffener MOI to preclude global shear buckling.
- This equation would appear to be redundant if your design already passes a global shear buckling check

Equation C10.8 $$I_v = 2.29\frac{d}{t}\left(\frac{Vh}{33E}\right)^{4/3}$$

Part 2: Bruhn Figure C10.9



$$\frac{I_v}{dt^3} = \frac{0.0217}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$$

Bruhn, page C10.8

A more recent criteria for stiffener stiffness (I_V) for both flat and curved webs is given by the curve in Fig. C10.9. When the stiffener is used purely as such and not as a means to transfer a concentrated external load to the beam web, the question arises as to what is the minimum number of fasteners required in attaching the stiffener to the web. For non-buckling webs, two criteria are suggested:-

(1) The stiffener should be attached to the flange at each end.

(2) The rivet pitch (spacing) should according to (Ref. 1) be at the most equal to 1/4 times the stiffener spacing, or 1/4 the web height if this is smaller, in order to justify the assumption of simple support at the edges of the web panel. Normal practice uses more rivets.

Derivation of Bruhn Figure C10.9

- Bruhn calls this the “more recent” method
- It appears in the 1965 and 1973 editions, but not in the 1942 or 1949 editions
- It is attributed to Chance Vought Corp. (label in lower right hand corner of Figure C10.9)

Figure C10.9

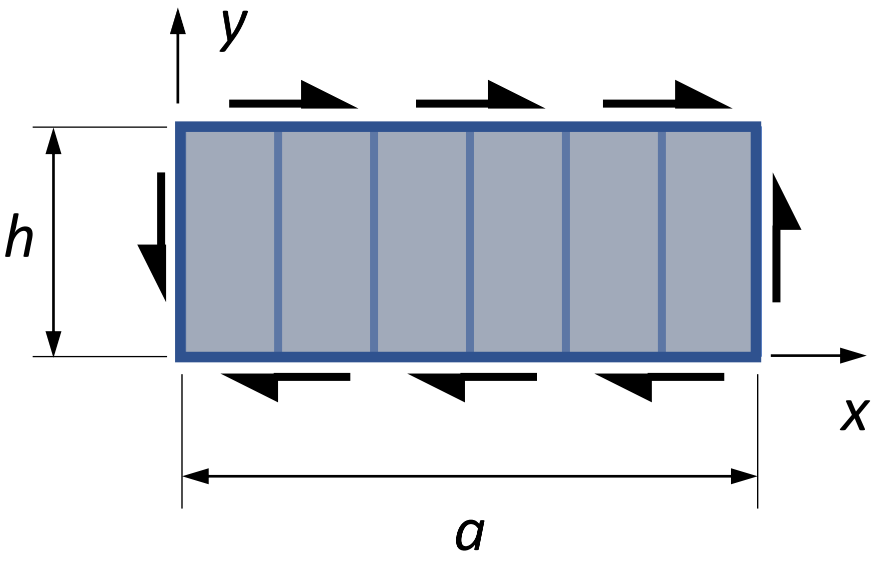
$$\frac{I_v}{dt^3} = \frac{0.0217}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$$

Derivation of Bruhn Figure C10.9

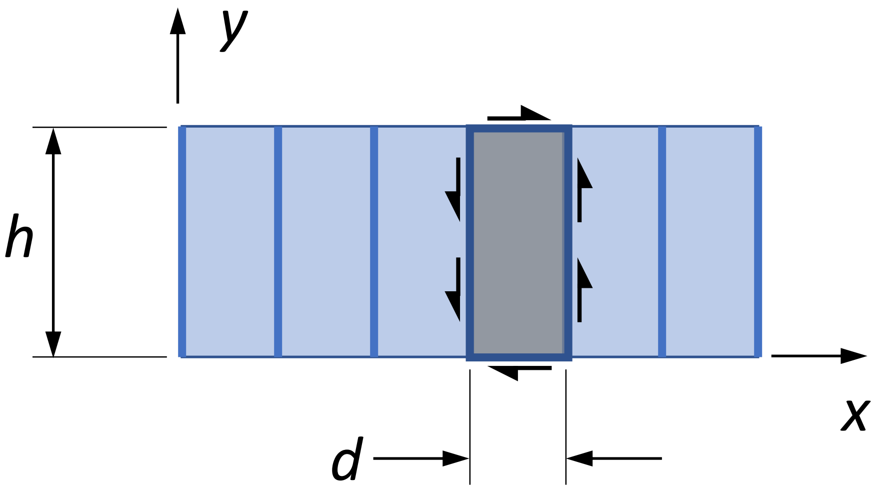
- Figure C10.9 appears to have been derived by equating the global shear buckling of a stiffened panel to the local shear buckling of the web between stiffeners

Equate these

Global Shear Buckling



Local Shear Buckling



Global Shear Buckling of Stiffened Panel

Seydel/Wagner buckling equation:

$$S = \frac{33E}{h}\left[\frac{t^3}{12}\left(\frac{I_v}{d}\right)^3\right]^{1/4}$$

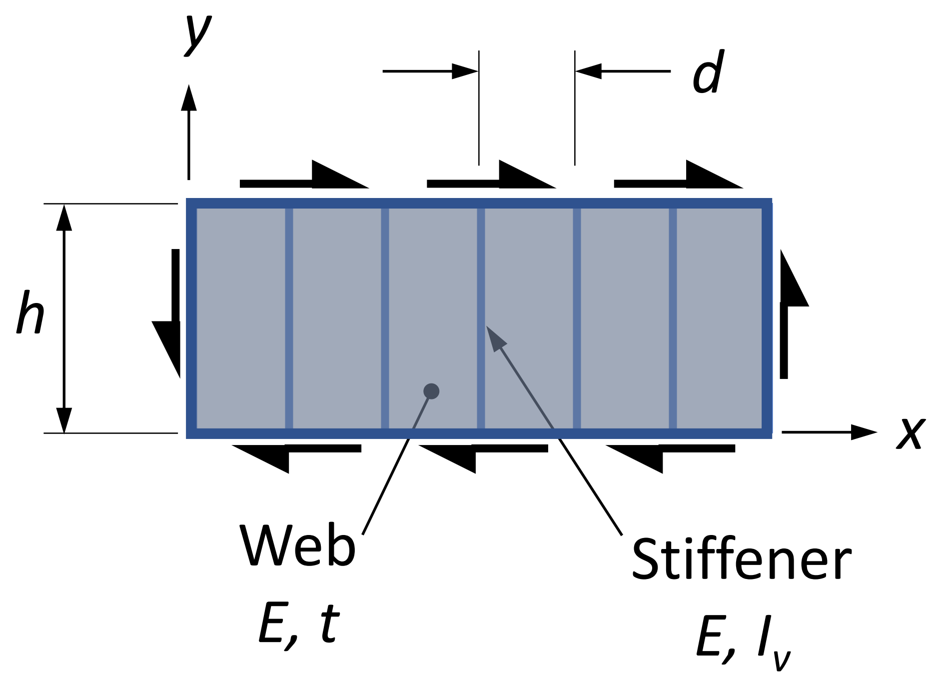
S = shear force (lb)

Convert to shear stress (psi):

$$F_{scr}^{global} = \frac{33}{h^2 t} E\left[\frac{t^3}{12}\left(\frac{I_v}{d}\right)^3\right]^{1/4}$$

Where *d* is the spacing and *h* is the height

Global Shear Buckling



Local Shear Buckling of Web Between Stiffeners

$$\tau_{cr} = \frac{k_s \pi^2 E}{12(1-\nu^2)}\left(\frac{t}{d}\right)^2$$

k_s = function of boundary conditions and aspect ratio d/h (d = short side)

$$\tau_{cr} = K_s E \left(\frac{t}{d}\right)^2$$

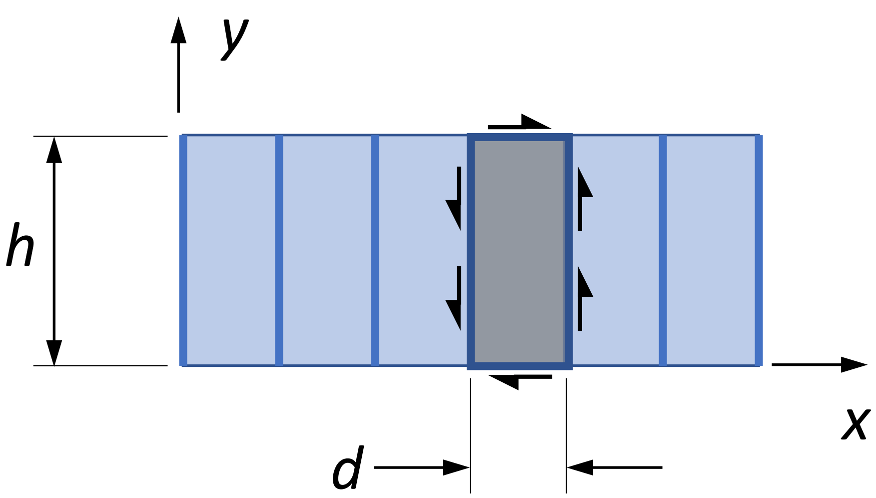
where: $$K_s = \frac{k_s \pi^2}{12(1-\nu^2)}$$

Note the difference between "big K" and "little k"

$$F_{scr}^{local} = \frac{E}{\left(\frac{d}{t\sqrt{K_s}}\right)^2}$$

Where d is the short side

Local Shear Buckling



$$d \leq h$$

Shear Buckling Coefficient

$$\tau_{cr} = \frac{k_s \pi^2 E}{12(1-\nu^2)} \left(\frac{t}{b}\right)^2$$

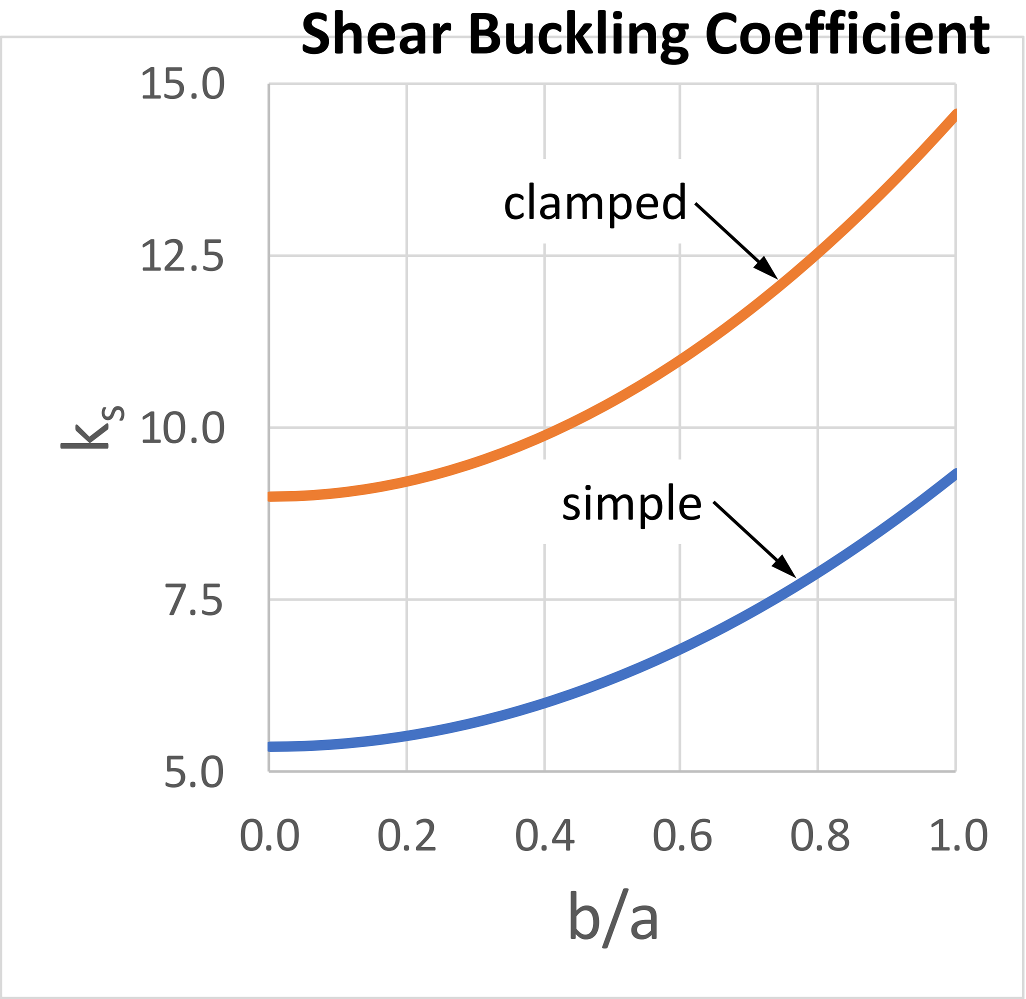
k_s = function of boundary conditions and aspect ratio

Parabolic approximations for k_s from Galambos (1998):

$$k_s = 5.34 + 4.0 \left(\frac{b}{a}\right)^2$$ All edges simply-supported

$$k_s = 8.98 + 5.6 \left(\frac{b}{a}\right)^2$$ All edges clamped

$b \leq a$ *b* is the short side



Derivation of Bruhn Figure C10.9

Set $F_{scr}^{global} = F_{scr}^{local}$ ⇨ $$\frac{33}{h^2 t} E \left[\frac{t^3}{12} \left(\frac{I_v}{d} \right)^3 \right]^{1/4} = \frac{E}{\left(\frac{d}{t\sqrt{K_s}} \right)^2}$$

RHS: *d* is the spacing and also the short side

LHS: *d* is the spacing and *h* is the height

$$t^{3/4} \frac{I_v^{3/4}}{d^{3/4}} = \frac{12^{1/4}}{33} \frac{h^2 t}{\left(\frac{d}{t\sqrt{K_s}} \right)^2}$$

$$t^{3/4} \frac{I_v^{3/4}}{d^{3/4}} = \frac{12^{1/4}}{33} \frac{t^3}{\left(\frac{d}{h\sqrt{K_s}} \right)^2}$$

Derivation of Bruhn Figure C10.9

$$\frac{I_v^{3/4}}{t^{9/4}d^{3/4}} = \frac{12^{1/4}}{33}\frac{1}{\left(\frac{d}{h\sqrt{K_s}}\right)^2}$$

$$\frac{I_v}{dt^3} = \frac{12^{1/3}}{33^{4/3}}\frac{1}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$$

$$\Rightarrow \quad \frac{I_v}{dt^3} = \frac{0.0216}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$$

This is Bruhn Fig. C10.9, except for tiny difference in the constant

$12^{1/3}$ / $33^{4/3}$ = 0.0216, Bruhn has 0.0217

Recap of Bruhn Figure C10.9

- This criterion equates global and local shear buckling and solves for stiffener moment of inertia in terms of other parameters
- Again, this MOI will not necessarily act as a panel breaker or enforce straight stiffeners, it simply gives the stiffener MOI required to force global and local shear buckling to occur simultaneously
- Again, this equation appears to be redundant if your particular design has already been shown to be good for both global and local shear buckling

Figure C10.9 $$\frac{I_v}{dt^3} = \frac{0.0217}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}} \qquad d \leq h$$

Summary

Bruhn's Stiffener Moment of Inertia Equations:

Equation C10.8 $$I_v = 2.29\frac{d}{t}\left(\frac{Vh}{33E}\right)^{4/3}$$

This is the I_v required to suppress global shear buckling at load V

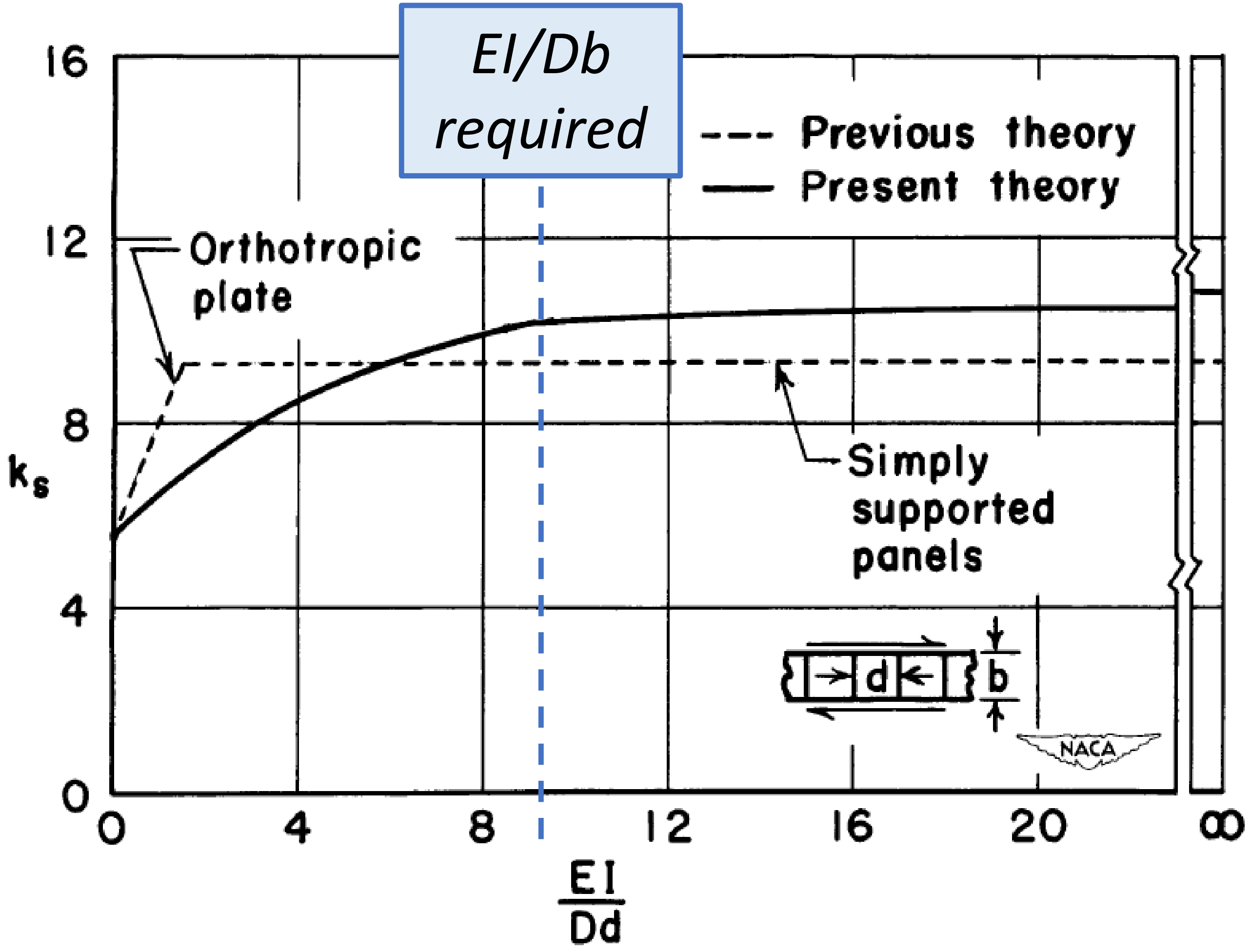
Figure C10.9 $$\frac{I_v}{dt^3} = \frac{0.0217}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$$

This is the I_v required to force global and local shear buckling to occur at the same load

Comments

- The purpose here was mainly to understand where the Bruhn moment of inertia formulas came from.
- Whether the Bruhn formulas are the right ones to use in your case is a different question, not taken up here…
- It is also worth noting that other stiffener MOI criteria exist. One is briefly discussed on the following pages…

Stiffened Shear Panels (Stein & Fralich)



- Stein & Fralich (1949) computed shear buckling loads for transversely stiffened panels
- The buckling coefficients were plotted vs. the EI per unit length of the stiffener compared to the bending stiffness D of the plate (EI/Dd)
- At some value of EI/Db, the buckling coefficient reaches a maximum value
- These transition values of EI/Db can be taken as the definition of the required stiffener MOI

Bleich Formula

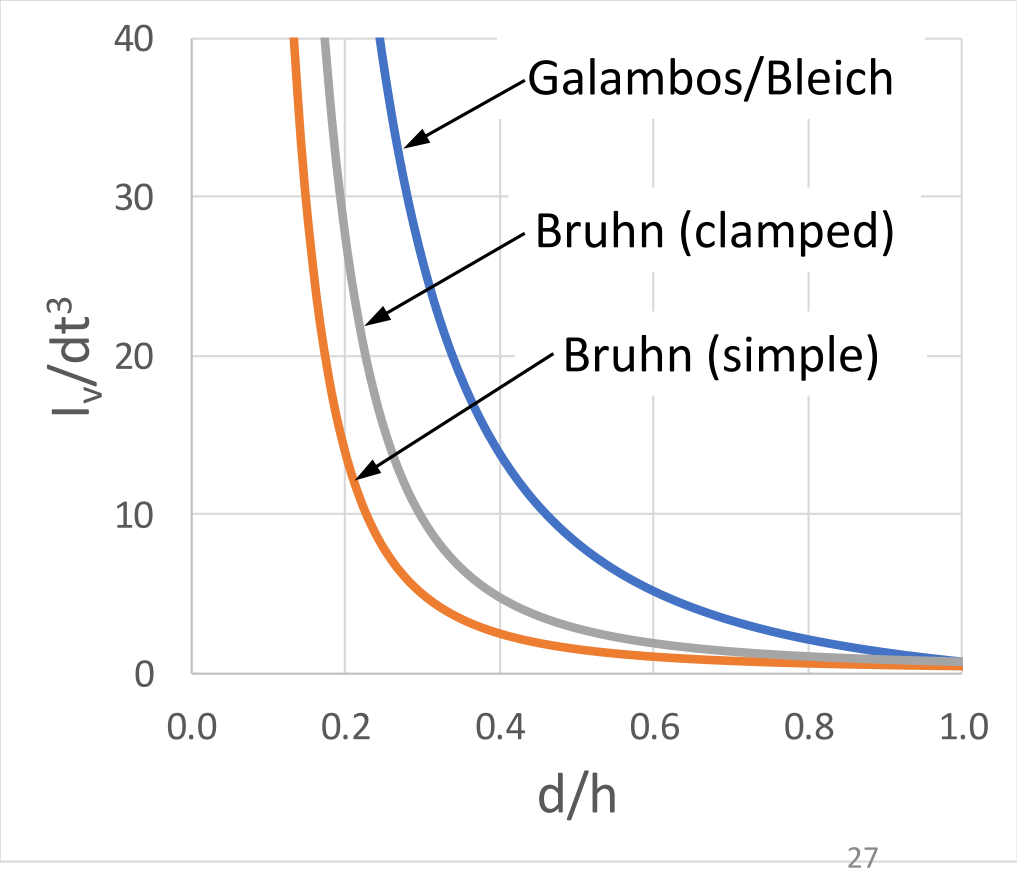
- Bleich (1952) used Stein & Fralich's results to determine a formula for the required stiffener moment of inertia. Bleich's formula is presented in the book by Galambos (1998):

Galambos Eqn. 6.36: $I = 2.5ht^3\left(\frac{h}{d} - 0.7\frac{d}{h}\right)$

Re-write as: $\frac{I}{dt^3} = 2.5\left[\left(\frac{h}{d}\right)^2 - 0.7\right]$

Compare to Bruhn Fig. C10.9: $\frac{I_v}{dt^3} = \frac{0.0217}{\left(\frac{d}{h\sqrt{K_s}}\right)^{8/3}}$

Take K_s in Bruhn as (1) all sides simply supported and (2) all sides clamped



Comments

- The Galambos/Bleich formula requires a greater stiffener MOI than Bruhn
  - For a stiffener spacing equal to half the web height ($d/h = 0.5$):
    - Galambos gives $I/dt^3 = 8.3$
    - Bruhn (Fig. C10.9) gives 1.5 (simple) or 2.8 (clamped)
    - That's a significant difference!
- The Bruhn criteria does not explicitly enforce a straight stiffener
  - The derivations do not put a limit on stiffener deflection; they impose constraints on the buckling load(s)
  - Undocumented FEM studies showed that while stiffeners sized to the Bruhn criterion increase the buckling load over an unstiffened web, the stiffeners do not remain straight; they bow along with the web
  - In those same FEM studies, the Galambos/Bleich criteria came much closer to providing stiffeners that remain straight during buckling

Bruhn Books

- Airplane Structural Design
  - 1942, 348 pages (earliest version)
- Analysis and Design of Flight Vehicle Structures
  - 1973, 979 pages (latest version)
- Analysis and Design of Missile Structures
  - by Bruhn, Orlando, & Myers, 1967
- Bruhn Supplement
  - by William F. McCombs, 1998
- Bruhn Errata
  - by Bill Gran, 2008

    http://www.grancorporation.com/Bruhn_Errata_2nd_Edition_Draft2.pdf

Figure 1.12
Professor Elmer F. Bruhn
(1899–1984).

From: "*One Small Step: The History of Aerospace Engineering at Purdue University*"

References

1. Bleich, Buckling Strength of Metal Structures, 1952
2. Bruhn, Airplane Structural Design, 1942
3. Bruhn, Analysis and Design of Flight Vehicle Structures, 1973
4. Galambos, Guide to Stability Design Criteria for Metal Structures, Fifth Edition, 1998
5. Jones, Mechanics of Composite Materials, 1975
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7. Stein & Fralich, Critical Shear Stress of Infinitely Long, Simply Supported Plate with Transverse Stiffeners, NACA-TN-1851, 1949
8. Timoshenko, Theory of Elastic Stability, Second Edition, 1961
9. Wagner, Sheet-Metal Airplane Construction, ASME Paper AER-53-18, 1930