The Johnson Parabola: A Brief Historical Note

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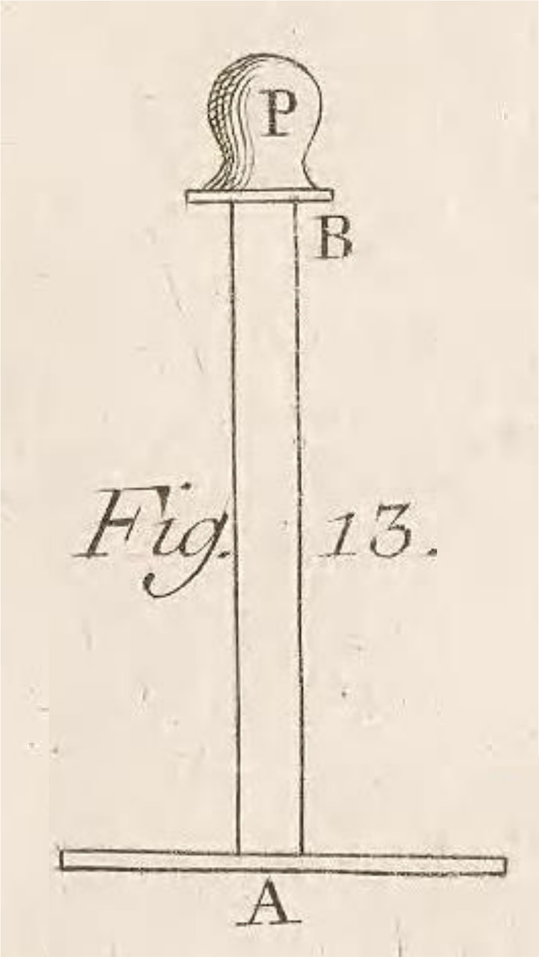
Euler's Column Formula

- Euler published his studies on elastic beams and columns in 1744 and 1759
- He derived a formula for the load *P* at which a uniform section, axially loaded column becomes unstable

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$



Convert to Stress

- Engineers like to deal with stress, so divide P_{cr} by A in Euler's formula
- We can also use an effective length L' (or a restraint coefficient c) to handle different boundary conditions

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2}$$

use $\rho^2 = \frac{I}{A}$

$$\sigma_{cr} = \frac{\pi^2 E}{\left(L/\rho\right)^2}$$

where: ρ = radius of gyration
L/ρ = slenderness ratio

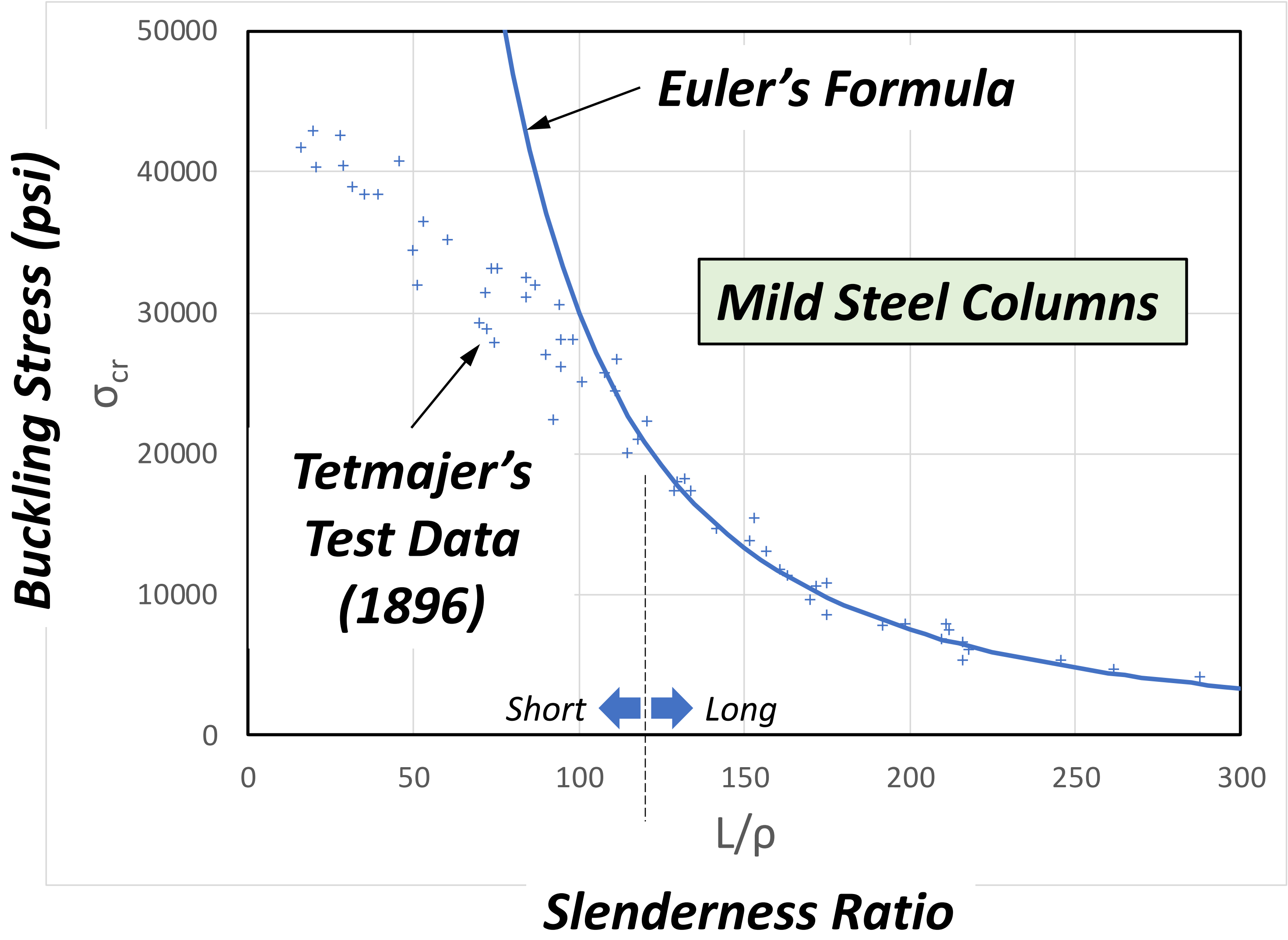
define $L' = \frac{L}{\sqrt{c}}$ where: L' = effective length
c = restraint coefficient

$$\sigma_{cr} = \frac{\pi^2 E}{\left(L'/\rho\right)^2}$$

$$\sigma_{cr} = \frac{c\pi^2 E}{\left(L/\rho\right)^2}$$

c = 1 → both ends simply-supported
c = 4 → both ends clamped

Comparison to Test Data



- Engineers found that test data did not agree with Euler's formula for "short" columns
- Engineers developed many alternate formulas for design purposes to handle the short column range
- We will only consider Johnson's parabola here

John Butler Johnson

JOHNSON, John Butler (1850–1902). An American civil engineer and educator. He was born at Marlboro, Ohio, and graduated at the University of Michigan in 1878. He was a member of the United States Lake and Mississippi River Surveys until 1883; professor of civil engineering at Washington University, St. Louis, Mo., from that date until 1898, and dean of the department of mechanics and engineering at the University of Wisconsin from 1898 until his death. The parabolic column formula, which he proposed, bears his name; his name is also connected with the roller extensometer, an instrument used in measuring the stretch in materials under test. In 1884 he assumed charge of the index department of the *Journal of the Association of Engineering Societies*, and in 1891 was put in charge of the timber-testing laboratory at St. Louis by the United States Forestry Bureau. His publications include *The Materials of Construction*, first published in 1897; *Theory and Practice of Surveying* (8th ed., 1904); *Engineering Contracts and Specifications* (3d ed., 1904); and he was joint author of *Modern Framed Structures* (1893).

"Johnson could teach all he knew and more."

Johnson's Parabolic Formula

- In his 1893 book, J. B. Johnson proposed a parabola for the short column range:

THE THEORY AND PRACTICE

OF

MODERN FRAMED STRUCTURES.

NEW YORK:

JOHN WILEY & SONS,

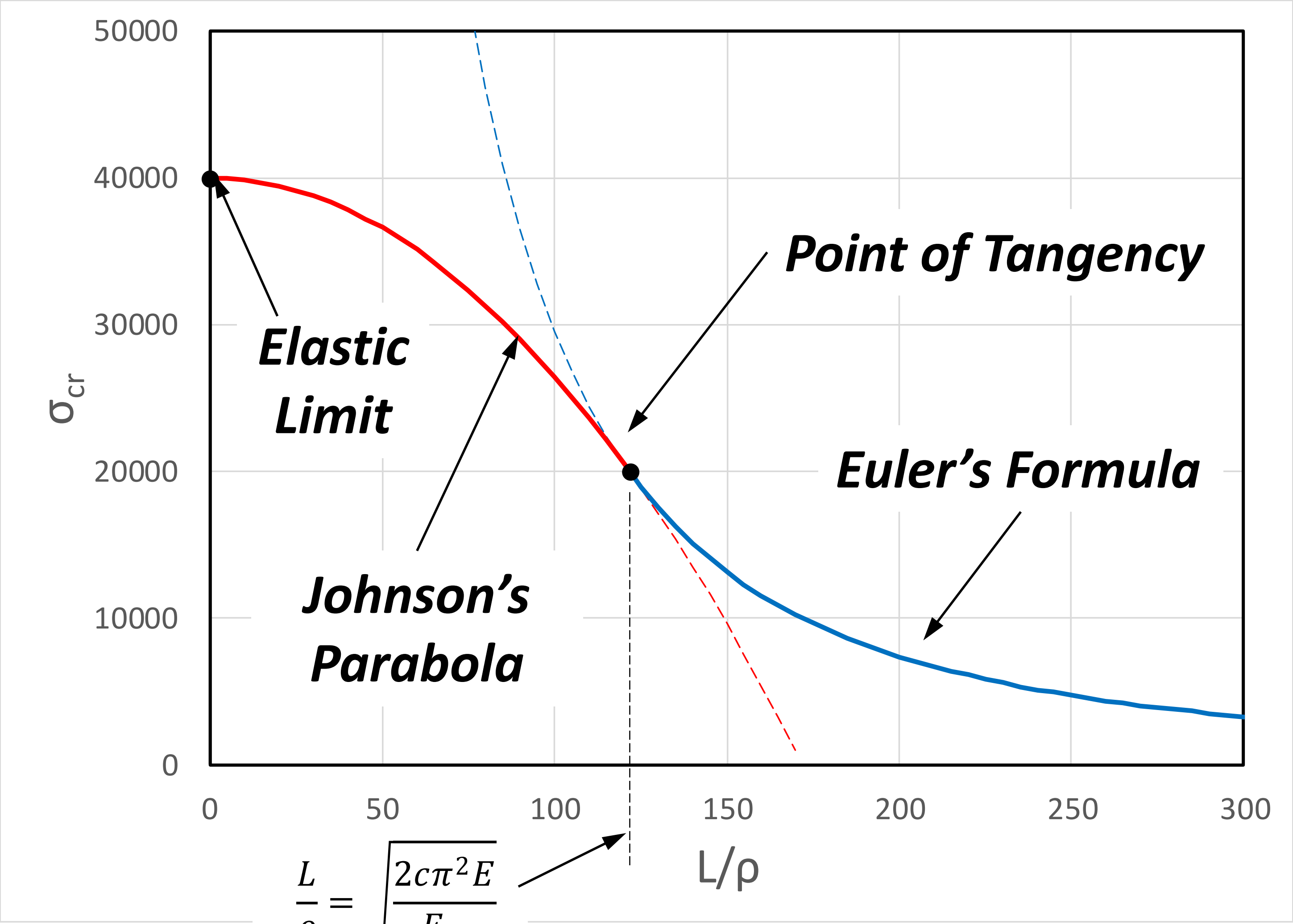
53 EAST TENTH STREET.

1893.

Page 148: **A New Formula.** $p = f - b\left(\frac{l}{r}\right)^2.$

To find the Equation of the Parabola having its Vertex at the Elastic Limit on the Axis of Loads, and Tangent to Euler's Curve.

Johnson’s Parabolic Formula



$$\frac{L}{\rho} = \sqrt{\frac{2c\pi^2 E}{F_{cy}}}$$

- The vertex of the parabola (at $L/\rho = 0$) was to be at the Elastic Limit of the material

$$p = f - b\left(\frac{L}{r}\right)^2.$$

- The constant b was computed to make the parabola tangent to Euler’s formula

“*Materials of Construction*”, J. B. Johnson, 1897

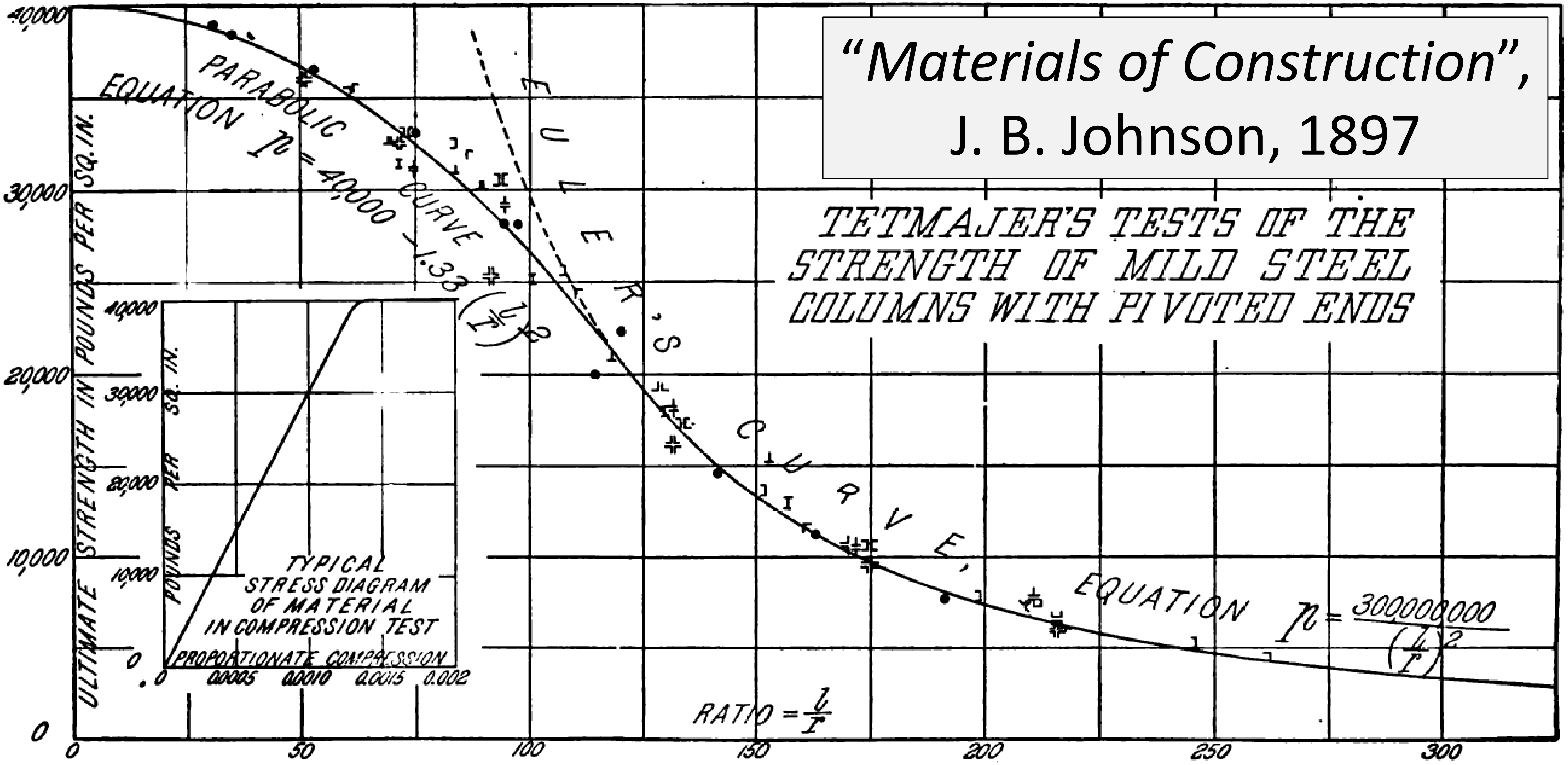


FIG. 298.—The Author's Parabolic Column Formula fitted to Tetmajer's Tests of Steel Columns. (*Communications*, vol. IV.)

Johnson’s Parabolic Formula

- Johnson’s k would be our $c\pi^2 E$
- For f (elastic limit), we would use F_{cy} (compressive yield strength)
- This equation is for primary instability of compact cross-sections, i.e. those that are not susceptible to local buckling before failure (see later note)

Eq. Tang. Parabola, $$p = f - \frac{f^2}{4k}\left(\frac{l}{r}\right)^2. \qquad (19)$$

Eq. Euler's Curve, $$p = \frac{k}{\left(\frac{l}{r}\right)^2}. \qquad (20)$$

Use on Airplane Structures

- Johnson’s Parabola has been used for the structural design of airplanes almost from the beginning of flight
- see following pages for a few examples…

1920 U.S. Army Air Service Handbook

Structural Analysis and Design
OF
Airplanes

ENGINEERING DIVISION
McCOOK FIELD
DAYTON, OHIO

June, 1920

The column formula that agrees best with test data on columns, whose L/ρ is less than the lower limit to which the Euler formula is applicable, is J. B. Johnson's parabolic formula,

$$P/A = f - \frac{f^2}{4\,c\pi^2 E}(L/\rho)^2$$

f = Yield point of material
c = A constant with same values as in Euler's formula.

1920 U.S. Army Air Service Handbook

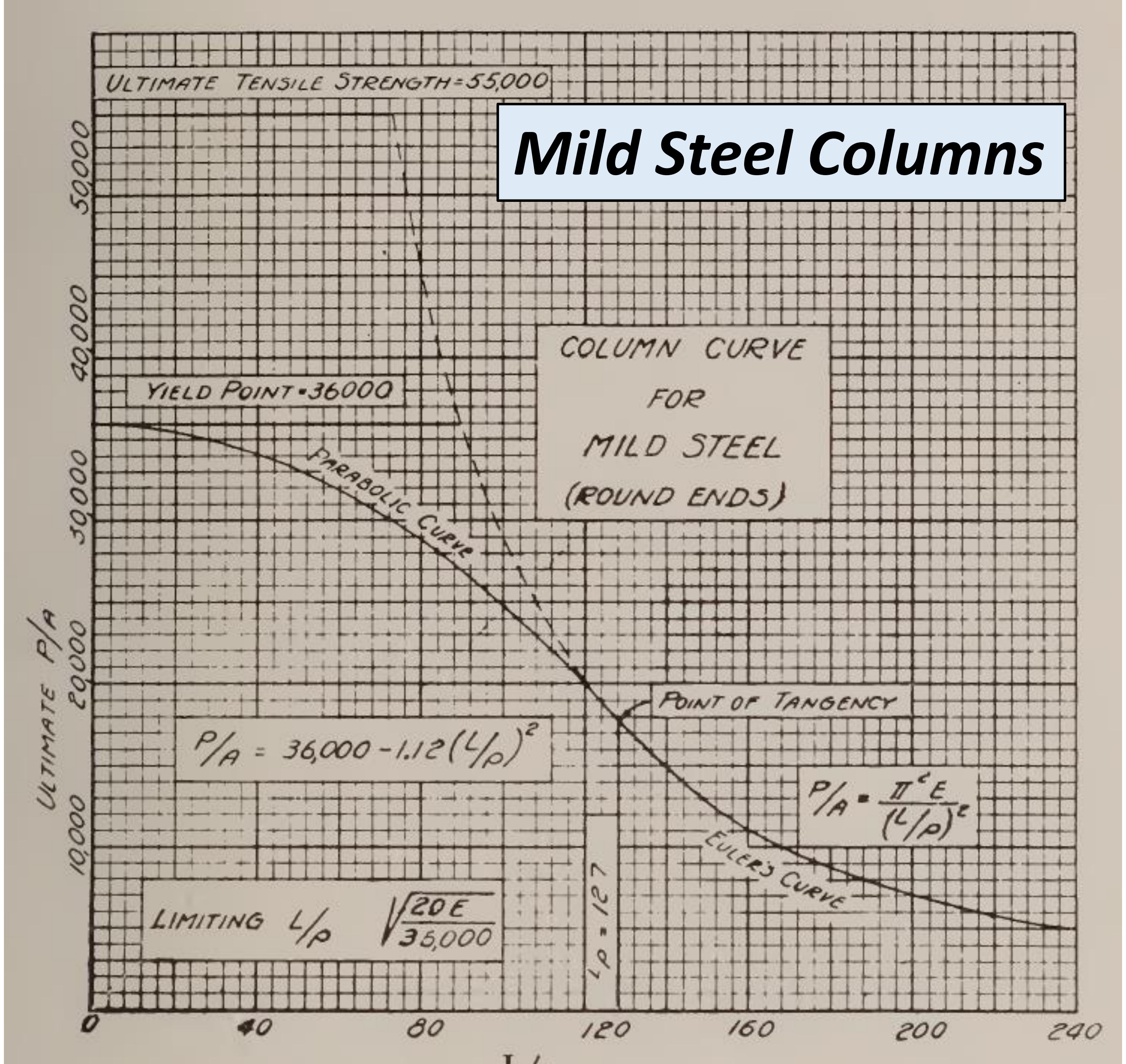


Fig. 25.

1920 U.S. Army Air Service Handbook

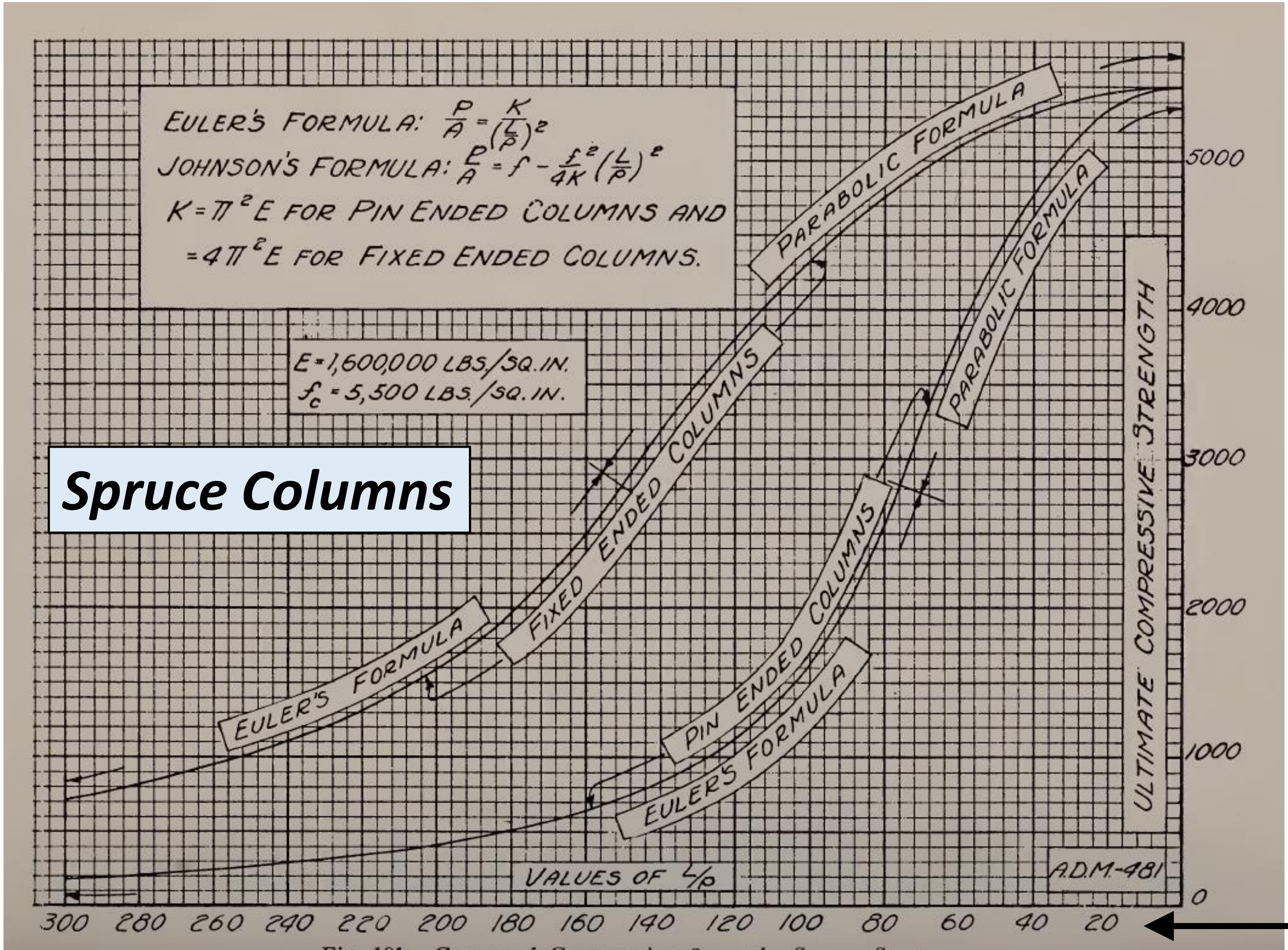


Fig. 181. Curves of Compressive Strength—Spruce Struts.

Note they plotted L/ρ increasing to the left!

ANC-5 (Predecessor to MIL-HDBK-5)

ANC–5

STRENGTH OF AIRCRAFT ELEMENTS

Issued by the

ARMY-NAVY-COMMERCE

COMMITTEE ON AIRCRAFT REQUIREMENTS

UNITED STATES

GOVERNMENT PRINTING OFFICE

WASHINGTON : 1938

$$F_c = F_{co}\left[1 - \frac{F_{co}(L'/\rho)^2}{4\pi^2 E}\right]$$ (2.0 parabola - Johnson formula)

F_c - allowable compressive stress.

F_{co} - column yield stress. (upper limit of column stress for primary failure)

ANC-5
1938 Edition

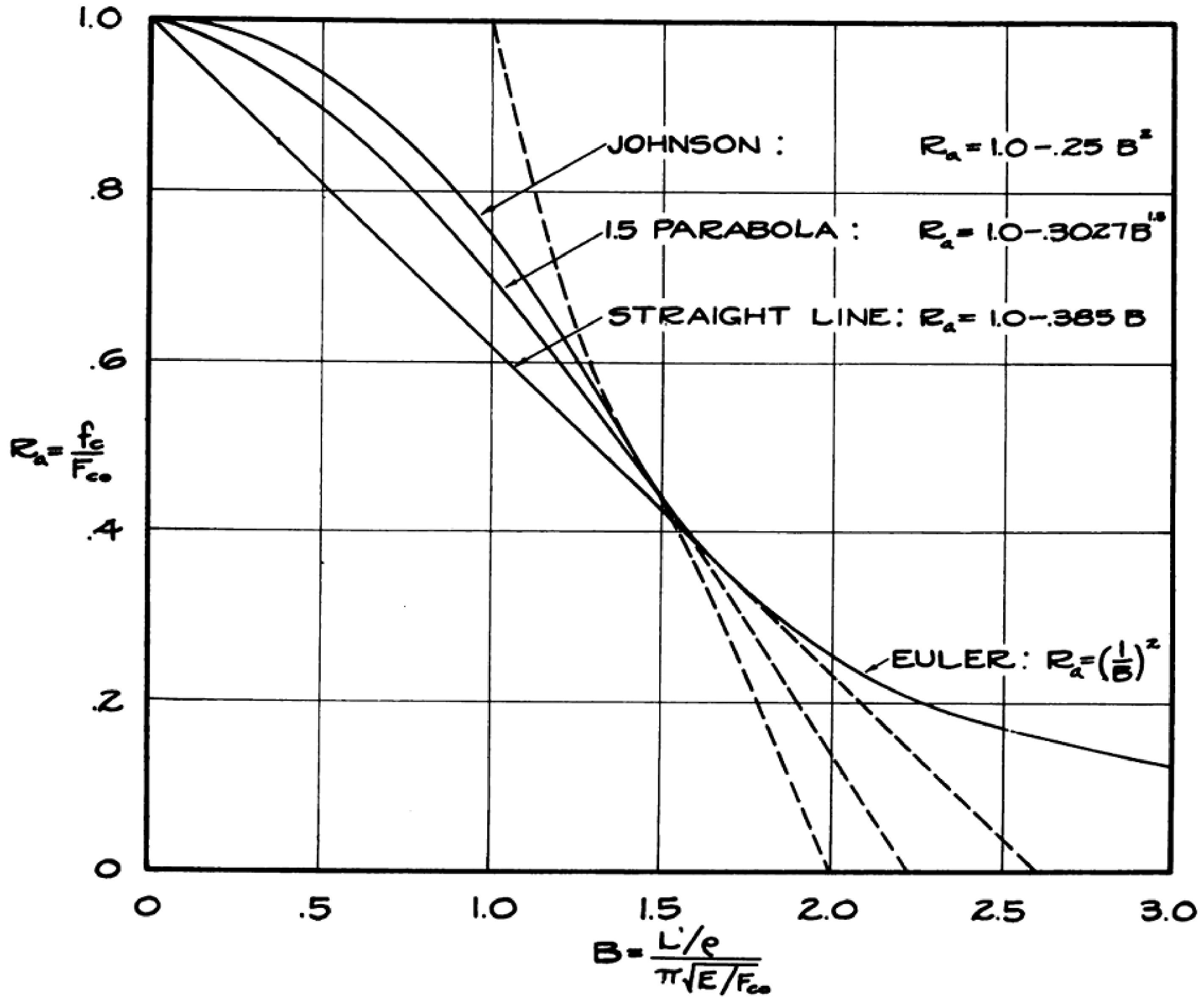


FIG. 1-3. VARIOUS COLUMN CURVES IN NON-DIMENSIONAL FORM

Thin-Walled Sections

- For thin-walled sections, Johnson's parabola can be used, but in this case the allowable stress at $L'/\rho = 0$ would be the crippling strength of the section F_{cc} (see Rivello, section 16-4)

<u>*Compact Section*</u>

$$F_c = F_{cy}\left[1 - \frac{F_{cy}}{4\pi^2 E}\left(\frac{L'}{\rho}\right)^2\right]$$

Yield strength of material

<u>*Thin-Walled Section*</u>

$$F_c = F_{cc}\left[1 - \frac{F_{cc}}{4\pi^2 E}\left(\frac{L'}{\rho}\right)^2\right]$$

Crippling strength of section

References (most are on internet)

- Euler, De Curvis Elasticis, 1744
- Euler, Sur la force des colonnes, 1759
- Johnson, Theory and Practice of Modern Framed Structures, Wiley, 1893
- Tetmajer, Die Gesetze der Knickungsfestigkeit der technisch wichtigsten Baustoffe, 1896
- Johnson, The Materials of Construction, Wiley, 1897
- Structural Analysis and Design of Airplanes, Second Edition, U.S. Air Service, 1920
- ANC-5, Strength of Structural Elements, Army Navy Commerce Committee, 1938
- Rivello, Theory and Analysis of Flight Structures, McGraw-Hill, 1969